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Proportionality, equality, and duality in bankruptcy problems with nontransferable utility

B. Dietzenbacher[1] · **A. Estévez-Fernández**[2] · **P. Borm**[3] · **R. Hendrickx**[3]

Abstract
This paper studies bankruptcy problems with nontransferable utility as a generalization of bankruptcy problems with monetary estate and claims. Following the theory on TU-bankruptcy, we introduce a duality notion for NTU-bankruptcy rules and derive several axiomatic characterizations of the proportional rule and the constrained relative equal awards rule.

Keywords NTU-bankruptcy · Duality · Proportional rule · Constrained relative equal awards rule

JEL Classification C79 · D63 · D74

1 Introduction

In a bankruptcy problem with transferable utility (cf. O'Neill 1982), claimants have individual claims on an perfectly divisible but insufficient estate. Bankruptcy theory analyzes allocations of the estate among the claimants, taking into account their claims. Bankruptcy problems with transferable utility are well-studied both from an axiomatic as well as a game theoretic perspective (cf. Thomson 2003, 2013, 2015).

Carpente et al. (2013) analyzed bankruptcy problems where claimants may have different and nonlinear utility functions. In particular, they put forward the channel assignment problem in wireless telecommunication networks. Other situations that fit into this framework are a company's pension fund which cannot meet its liabilities and employees experience their losses differently due to wealth distinctions, a university allocating its restricted budget and departments experience their deficits differently due to equipment distinctions, or a

✉ B. Dietzenbacher
bdietzenbacher@hse.ru

1 International Laboratory of Game Theory and Decision Making, National Research University Higher School of Economics, St. Petersburg, Russian Federation

2 Tinbergen Institute and Department of Econometrics and Operations Research, VU University Amsterdam, Amsterdam, The Netherlands

3 CentER and Department of Econometrics and Operations Research, Tilburg University, Tilburg, The Netherlands

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supranational organization which limits polluting emissions and countries experience their reductions differently due to industry distinctions. This information is relevant for fair allocations and it can be taken into account by modeling these situations as bankruptcy problems with nontransferable utility. Alternatively, bankruptcy problems with nontransferable utility can be interpreted as bargaining problems with claims (cf. Chun and Thomson 1992) or Nash rationing problems (cf. Mariotti and Villar 2005).

This paper studies bankruptcy problems with nontransferable utility as a generalization of bankruptcy problems with monetary estate and claims. The estate is the set of attainable utility allocations which are assumed to be normalized such that allocating nothing to a claimant corresponds to zero utility. The claims represent the individual utility claims of the claimants on the estate. Orshan et al. (2003), Carpente et al. (2013), Dietzenbacher (2018) and Estévez-Fernández et al. (2020) analyzed NTU-bankruptcy problems from a game theoretic perspective by defining an associated bankruptcy game and focusing on the structure of the core. We axiomatically approach NTU-bankruptcy problems and especially show that bankruptcy theory can be extended by adequately reformulating rules and properties, with a focus on the proportional rule, the constrained relative equal awards rule, and duality.

The proportional rule for bankruptcy problems prescribes the efficient allocation which is proportional to the vector of claims. The constrained equal awards rule for TU-bankruptcy problems divides the estate as equally as possible under the condition that claimants are not allocated more than their claims. Interpreting the utopia values, the maximal individual payoffs within the estate, as aspirational references inducing an egalitarian direction, we define the constrained *relative* equal awards rule for NTU-bankruptcy problems which allocates payoffs as relatively equally as possible, i.e. proportional to the utopia values. This generalizes the constrained equal awards rule for TU-bankruptcy problems while ensuring covariance under individual rescaling of utility.

Within TU-bankruptcy theory, duality of rules and many defined properties are based on modifications of the estate. Such a modification can be interpreted as a homothetic transformation which preserves the shape of the estate. In line with this interpretation, we apply a uniform scaling operation to modify the estate of NTU-bankruptcy problems. It is important to note that uniform scaling is not in conflict with individual scaling, i.e. all considered notions are covariant under individual rescaling of utility. On the one hand, we define duality of NTU-bankruptcy rules and show that the proportional rule is self-dual and that the constrained relative equal awards rule is dual to the constrained relative equal losses rule. On the other hand, we define several properties concerning shape-preserving changes in the estate and derive corresponding axiomatic characterizations. In particular, we generalize the characterizations of the proportional rule based on estate linearity (cf. Chun 1988), and composition down/up and self-duality (cf. Young 1988). Moreover, we generalize the characterizations of the constrained relative equal awards rule based on symmetry, truncation invariance, and composition up (cf. Dagan 1996), claim monotonicity and conditional full compensation (cf. Yeh 2006), and conditional full compensation and composition down (cf. Herrero and Villar 2002; Yeh 2004). Interestingly, we show that the constrained relative equal awards rule also shares a characteristic feature with the serial cost sharing rule (cf. Moulin and Shenker 1992) by generalizing a characterization based on symmetry and independence of larger claims.

This paper is organized in the following way. Section 2 formally introduces NTU-bankruptcy problems and NTU-bankruptcy rules. In Sects. 3 and 4, we introduce a duality notion and axiomatically study NTU-bankruptcy rules, respectively. Section 5 concludes.

2 Bankruptcy problems with nontransferable utility

Let N be a nonempty and finite set of *claimants*. For any $x, y \in \mathbb{R}^N_+$, $x \leq y$ denotes $x_i \leq y_i$ for all $i \in N$, and $x < y$ denotes $x_i < y_i$ for all $i \in N$. The zero-vector $x \in \mathbb{R}^N_+$ with $x_i = 0$ for all $i \in N$ is denoted by 0_N.

A *bankruptcy problem with nontransferable utility* (*NTU-bankruptcy problem*, cf. Orshan et al. 2003) is a triple (N, E, c), where $E \subseteq \mathbb{R}^N_+$ is the *estate* which is assumed to be

- Nonempty, closed, and bounded;
- Comprehensive, i.e. $E = \{x \in \mathbb{R}^N_+ \mid \exists_{y \in E} : y \geq x\}$;
- Nonleveled, i.e. $P(E) = \{x \in E \mid \neg\exists_{y \in E, y \neq x} : y \geq x\} = \{x \in E \mid \neg\exists_{y \in E} : y > x\}$;[1]

and $c \in \mathbb{R}^N_+$ with $c \not< x$ for all $x \in E$ is the vector of *claims* of N on E. The estate is the set of attainable utility allocations which are assumed to be normalized such that allocating nothing to a claimant corresponds to zero utility. The claim vector represents the individual utility claims on the estate. Since our analysis is based on a fixed population, an NTU-bankruptcy problem is denoted by (E, c). Note that any TU-bankruptcy problem (cf. O'Neill 1982) with monetary estate $M \in \mathbb{R}_+$ induces an NTU-bankruptcy problem with $E = \{x \in \mathbb{R}^N_+ \mid \sum_{i \in N} x_i \leq M\}$. Let BR^N denote the class of all NTU-bankruptcy problems with claimants N, estate $E \neq \{0_N\}$, and claims $c \notin E$.

Let $(E, c) \in \mathrm{BR}^N$. For any $t \in \mathbb{R}_{++}$, the set $tE \subseteq \mathbb{R}^N_+$ is defined by $tE = \{tx \mid x \in E\}$. The vector of *utopia values* $u^E \in \mathbb{R}^N_{++}$ is defined by

$$u^E = (\max\{x_i \mid x \in E\})_{i \in N}\,.$$

Note that $u^{tE} = tu^E$ for all $t \in \mathbb{R}_{++}$. Let $x \in \mathbb{R}^N_+ \setminus \{0_N\}$. The scalar $\lambda^{E,x} \in \mathbb{R}_{++}$ is defined in such a way that

$$x \in \mathrm{P}\big(\lambda^{E,x} E\big) \quad \text{and} \quad \frac{1}{\lambda^{E,x}} x \in \mathrm{P}(E).$$

Note that the conditions on E imply that $\lambda^{E,x}$ uniquely exists. Moreover, $\lambda^{E,x} \leq 1$ if $x \in E$, and $\lambda^{E,x} > 1$ if $x \notin E$. For any $t \in \mathbb{R}_{++}$,

$$\lambda^{tE,x} = \frac{\lambda^{E,x}}{t} \quad \text{and} \quad \lambda^{E,tx} = t\lambda^{E,x}.$$

Note that $(tE, x) \in \mathrm{BR}^N$ for all $t \in (0, \lambda^{E,x})$, and $(E, tx) \in \mathrm{BR}^N$ for all $t \in (\frac{1}{\lambda^{E,x}}, \infty)$.

A *bankruptcy rule* f assigns to any bankruptcy problem (E, c) a payoff allocation $f(E, c) \in \mathrm{P}(E)$ for which $f(E, c) \leq c$. These conditions are known as *efficiency* and *claims boundedness*, respectively. Note that $f(E, c) = 0_N$ if $E = \{0_N\}$, and $f(E, c) = c$ if $c \in E$.

The *proportional rule*, Prop, assigns to any bankruptcy problem $(E, c) \in \mathrm{BR}^N$ the payoff allocation which is proportional to the vector of claims, i.e.

$$\mathrm{Prop}(E, c) = \frac{1}{\lambda^{E,c}} c.$$

[1] In other words, the strong Pareto set coincides with the weak Pareto set. This condition is also called *strict comprehensiveness* (see e.g. Roemer 1998).

The *constrained relative equal awards rule*, CREA, assigns to any bankruptcy problem $(E, c) \in \mathrm{BR}^N$ the payoff allocation which divides utility awards as relatively equal as possible, i.e.

$$\mathrm{CREA}(E, c) = \left(\min\{c_i, \alpha^{E,c} u_i^E\}\right)_{i \in N},$$

where $\alpha^{E,c} \in (0, 1)$ is such that $\mathrm{CREA}(E, c) \in \mathrm{P}(E)$.

The *constrained relative equal losses rule*, CREL, assigns to any bankruptcy problem $(E, c) \in \mathrm{BR}^N$ the payoff allocation which divides utility losses as relatively equal as possible, i.e.

$$\mathrm{CREL}(E, c) = \left(\max\{0, c_i - \beta^{E,c} u_i^E\}\right)_{i \in N},$$

where $\beta^{E,c} \in \mathbb{R}_{++}$ is such that $\mathrm{CREL}(E, c) \in \mathrm{P}(E)$.

For any $(E, c) \in \mathrm{BR}^N$ with $E = \{x \in \mathbb{R}_+^N \mid \sum_{i \in N} x_i \leq M\}$ induced by a TU-bankruptcy problem with monetary estate $M \in \mathbb{R}_{++}$, we have $u_i^E = M$ for all $i \in N$ and the proportional rule, the constrained relative equal awards rule, and the constrained relative equal losses rule coincide with the classic proportional rule, the constrained equal awards rule, and the constrained equal losses rule, respectively.

3 Duality

In this section, we introduce a duality notion for bankruptcy rules. Two TU-bankruptcy rules are dual (cf. Aumann and Maschler 1985) if one rule allocates awards in the same way as the other rule allocates losses. How a rule allocates losses is determined by applying the rule to the 'dual problem' in which the estate equals the total monetary deficit in the original problem. Alternatively, the estate of the dual problem can be obtained by scaling the original estate from the claims vector in the direction of the origin in such a way that the boundary intersects the assigned payoff allocation. The latter interpretation can also be applied in the NTU-bankruptcy context. Two NTU-bankruptcy rules are dual when one rule allocates utility awards in the same way as the other rule allocates utility losses in the dual problem obtained by this scaling method.

Dual Bankruptcy Rules
Two bankruptcy rules f and g are *dual* if for all $(E, c) \in \mathrm{BR}^N$,[2]

$$f(E, c) = c - g\left(\lambda^{E,c-f(E,c)} E, c\right) \text{ and } g(E, c) = c - f\left(\lambda^{E,c-g(E,c)} E, c\right).$$

The following lemma shows that a bankruptcy rule has at most one dual.

Lemma 1 *Let f, g, and h be bankruptcy rules. If f and g are dual, and f and h are dual, then $g = h$.*

Proof Assume that f and g are dual, and that f and h are dual. Let $(E, c) \in \mathrm{BR}^N$. Then

$$\begin{aligned} g(E, c) &= c - f\left(\lambda^{E,c-g(E,c)} E, c\right) = h\left(\lambda^{E,c-f\left(\lambda^{E,c-g(E,c)} E,c\right)} E, c\right) \\ &= h(\lambda^{E,g(E,c)} E, c) = h(E, c), \end{aligned}$$

[2] Note that $(\lambda^{E,c-f(E,c)} E, c) \in \mathrm{BR}^N$ since $\lambda^{E,c-f(E,c)} \in (0, \lambda^{E,c})$.

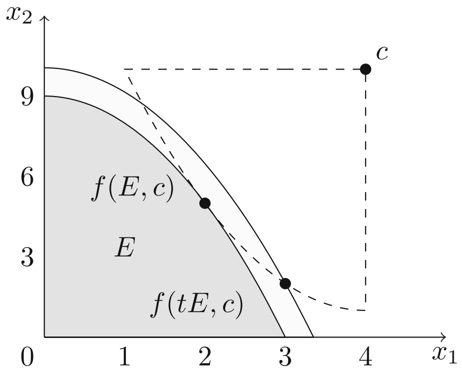
where the first and third equality follow from duality of f and g, the second equality follows from duality of f and h, and the last equality follows from $g(E,c) \in \mathrm{P}(E)$, which implies $\lambda^{E,g(E,c)} = 1$. Hence, $g = h$.

A bankruptcy rule is self-dual if it coincides with its dual.

Self-Dual Bankruptcy Rule
A bankruptcy rule f is *self-dual* if $f(E,c) = c - f(\lambda^{E,c-f(E,c)}E, c)$ for all $(E,c) \in \mathrm{BR}^N$.

In contrast to TU-bankruptcy, an NTU-bankruptcy rule does not necessarily have a dual, as the following example illustrates.

Example 1 Let $N = \{1,2\}$ and consider the bankruptcy problem $(E,c) \in \mathrm{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + x_2 \leq 9\}$ and $c = (4,10)$. Then, $\lambda^{E,c} = 2$ and $(1,8), (2,5) \in \mathrm{P}(E)$. Let $t \in (0, \lambda^{E,c})$ be given by $t = \frac{1}{9}(1+\sqrt{82})$ and let f be a bankruptcy rule such that $f(E,c) = (2,5)$ and $f(tE,c) = (3,2)$.



Suppose that g is a dual bankruptcy rule. Then

$$(2,5) = f(E,c) = c - g(\lambda^{E,c-f(E,c)}E,c) = c - g(\lambda^{E,(2,5)}E,c) = c - g(E,c)$$
$$\text{and } (3,2) = f(tE,c) = c - g(\lambda^{E,c-f(tE,c)}E,c) = c - g(\lambda^{E,(1,8)}E,c) = c - g(E,c).$$

This means that g is not a well-defined rule. Hence, f does not have a dual.

However, it is readily seen that all rules satisfying *path monotonicity*, as defined in Sect. 4, have a dual. In particular, this is the case for the proportional rule, the constrained relative equal awards rule, and the constrained relative equal losses rule. In fact, the proportional rule is self-dual and the constrained relative equal awards rule and the constrained relative equal losses rule are dual.

Theorem 1 (i) *The proportional rule is self-dual.*
(ii) *The constrained relative equal awards rule and the constrained relative equal losses rule are dual.*

Proof (i) Let $(E,c) \in \mathrm{BR}^N$. Then

$$\begin{aligned}\mathrm{Prop}(\lambda^{E,c-\mathrm{Prop}(E,c)}E,c) &= \frac{1}{\lambda^{\lambda^{E,c-\mathrm{Prop}(E,c)}E,c}}c = \frac{\lambda^{E,c-\mathrm{Prop}(E,c)}}{\lambda^{E,c}}c = \frac{\lambda^{E,\left(1-\frac{1}{\lambda^{E,c}}\right)c}}{\lambda^{E,c}}c \\ &= \left(1-\frac{1}{\lambda^{E,c}}\right)\frac{\lambda^{E,c}}{\lambda^{E,c}}c = c - \frac{1}{\lambda^{E,c}}c = c - \mathrm{Prop}(E,c).\end{aligned}$$

Hence, the proportional rule is self-dual.

(ii) Let $(E, c) \in \mathrm{BR}^N$. First, we show that $\mathrm{CREA}(E, c) = c - \mathrm{CREL}(\lambda^{E,c-\mathrm{CREA}(E,c)}E, c)$. Denote $d = \lambda^{E,c-\mathrm{CREA}(E,c)}$. Suppose that $d\beta^{dE,c} \leq \alpha^{E,c}$. For all $i \in N$,

$$\begin{aligned}\mathrm{CREL}_i(dE, c) &= \max\left\{0, c_i - \beta^{dE,c}u_i^{dE}\right\} = c_i - \min\{c_i, d\beta^{dE,c}u_i^E\}\\ &\geq c_i - \min\{c_i, \alpha^{E,c}u_i^E\} = c_i - \mathrm{CREA}_i(E, c).\end{aligned}$$

Since E is nonleveled, $\mathrm{CREL}(dE, c) \in \mathrm{P}(dE)$, and $c - \mathrm{CREA}(E, c) \in \mathrm{P}(dE)$, this means that $\mathrm{CREL}(dE, c) = c - \mathrm{CREA}(E, c)$. Clearly, similar arguments apply to the case $d\beta^{dE,c} > \alpha^{E,c}$. Second, we show that $\mathrm{CREL}(E, c) = c - \mathrm{CREA}(\lambda^{E,c-\mathrm{CREL}(E,c)}E, c)$. Denote $d' = \lambda^{E,c-\mathrm{CREL}(E,c)}$. Suppose that $d'\alpha^{d'E,c} \leq \beta^{E,c}$. For all $i \in N$,

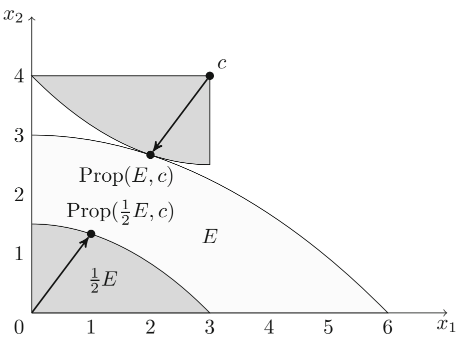
$$\begin{aligned}\mathrm{CREA}_i(d'E, c) &= \min\left\{c_i, \alpha^{d'E,c}u_i^{d'E}\right\} = c_i - \max\{0, c_i - d'\alpha^{d'E,c}u_i^E\}\\ &\leq c_i - \max\{0, c_i - \beta^{E,c}u_i^E\} = c_i - \mathrm{CREL}_i(E, c).\end{aligned}$$

Since E is nonleveled, $\mathrm{CREA}(d'E, c) \in \mathrm{P}(d'E)$, and $c - \mathrm{CREL}(E, c) \in \mathrm{P}(d'E)$, this means that $\mathrm{CREA}(d'E, c) = c - \mathrm{CREL}(E, c)$. Clearly, similar arguments apply to the case $d'\alpha^{d'E,c} > \beta^{E,c}$. Hence, CREA and CREL are dual.

Theorem 1 is illustrated by the following example. Note that the dual problem is not uniquely defined, but depends on the bankruptcy rule under consideration.

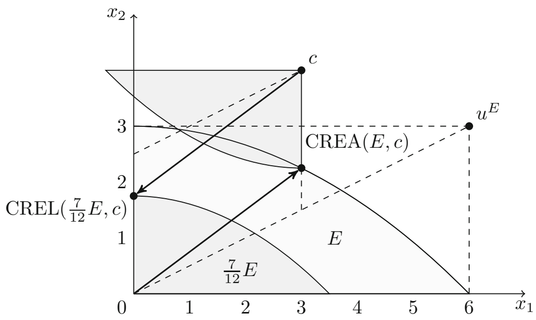
Example 2 Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36\}$ and $c = (3, 4)$. Then $\lambda^{E,c} = \frac{3}{2}$ and $\mathrm{Prop}(E, c) = \frac{2}{3}c = (2, 2\frac{2}{3})$. This means that $\lambda^{E,c-\mathrm{Prop}(E,c)} = \frac{1}{2}$. Since the proportional rule is self-dual,

$$\mathrm{Prop}(E, c) = c - \mathrm{Prop}(\tfrac{1}{2}E, c).$$



Moreover, $u^E = (6, 3)$ and $\alpha^{E,c} = \frac{3}{4}$. This means that $\mathrm{CREA}(E, c) = (3, 2\frac{1}{4})$ and $\lambda^{E,c-\mathrm{CREA}(E,c)} = \frac{7}{12}$. Since the constrained relative equal awards rule and the constrained relative equal losses rule are dual,

$$\mathrm{CREA}(E, c) = c - \mathrm{CREL}(\tfrac{7}{12}E, c).$$



4 Axiomatic characterizations

In this section, we axiomatically study NTU-bankruptcy rules. First, we focus on properties concerning changes in the estate. Instead of considering arbitrary changes in the estate, we restrict to homothetic transformations. This allows us to analyze the implications when the shape of the estate is preserved. The payoff path of a bankruptcy rule represents all prescribed payoff allocations when the estate is uniformly scaled. A bankruptcy rule satisfies path linearity when this payoff path is linear. Path composition down says that solutions on the payoff path can replace the claim vector when the estate is scaled down. Path composition up says that solutions on the payoff path can act as a new origin from which the estate is scaled up again. Path monotonicity and path continuity require that the payoff path is nondecreasing and continuous, respectively.

A bankruptcy rule f satisfies

- *Path linearity* if $f(tE, c) = tf(E, c)$ for all $t \in (0, 1)$;
- *Path composition down* if $f(tE, c) = f(tE, f(E, c))$ for all $t \in (0, 1)$;
- *Path composition up* if $f(E, c) = f(tE, c) + f(\lambda^{E, f(E,c)-f(tE,c)}E, c - f(tE, c))$ for all $t \in (0, 1)$;
- *Path monotonicity* if $f(tE, c) \leq f(E, c)$ for all $t \in (0, 1)$;
- *Path continuity* if $f(tE, c)$ is continuous in $t \in (0, 1)$.

As in TU-bankruptcy, path linearity implies both path composition down and path composition up, path composition down and path composition up both imply path monotonicity, and path monotonicity implies path continuity. This is summarized in the following diagram.

$$\begin{tikzcd}
 & \text{path linearity} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{path composition down} \arrow[dr, Rightarrow] & & \text{path composition up} \arrow[dl, Rightarrow] \\
 & \text{path monotonicity} \arrow[d, Rightarrow] & \\
 & \text{path continuity} &
\end{tikzcd}$$

Two properties for bankruptcy rules are *dual* (cf. Herrero and Villar 2001) if for any two dual bankruptcy rules, one property is satisfied by one rule if and only if the other property is satisfied by the other rule. A property for bankruptcy rules is *self-dual* if for any two dual bankruptcy rules, the property is satisfied by one rule if and only if it is satisfied by the other rule. As in TU-bankruptcy, path linearity is self-dual, path composition down and path composition up are dual, path monotonicity is self-dual, and path continuity is self-dual. This is captured by the following lemma.

Lemma 2 (i) *Path linearity is self-dual.*
(ii) *Path composition down and path composition up are dual.*
(iii) *Path monotonicity is self-dual.*
(iv) *Path continuity is self-dual.*

The proportional rule satisfies path linearity. The constrained relative equal awards rule and the constrained relative equal losses rule do not satisfy path linearity, but they do satisfy path composition down and path composition up. This is summarized in the following table.

	Prop	CREA	CREL
Path linearity	+	−	−
Path composition down	+	+	+
Path composition up	+	+	+
Path monotonicity	+	+	+
Path continuity	+	+	+

Inspired by Chun (1988) and Young (1988), we generalize the axiomatic characterizations of the proportional rule in terms of path linearity; path composition down and self-duality; and path composition up and self-duality to the domain of NTU-bankruptcy problems. The proof is postponed to the "Appendix".

Theorem 2 *The proportional rule is the unique bankruptcy rule satisfying*

(i) *Path linearity;*
(ii) *Path composition down and self-duality;*
(iii) *Path composition up and self-duality.*

Next, we focus on properties related to the constrained relative equal awards rule. A bankruptcy rule satisfies relative symmetry if claimants with relatively equal claims get relatively equal payoffs. This generalizes the equal treatment of claimants with equal claims property for TU-bankruptcy rules while ensuring covariance under individual rescaling of utility. Claim monotonicity states that a claimant cannot be worse off when its claim increases. Truncation invariance requires that the allocated payoffs only depend on the truncated claims, i.e. the claims truncated by the utopia values. Conditional full compensation imposes that claimants with small enough relative claims are fully reimbursed. A relative claim is considered small enough if all claimants could be fully reimbursed when their relative claims are truncated by this specific relative claim. Independence of larger relative claims implies that a claimant's payoff does not depend on the larger relative claims.

A bankruptcy rule f satisfies

- *Relative symmetry* if $\frac{f_i(E,c)}{u_i^E} = \frac{f_j(E,c)}{u_j^E}$ for all $i, j \in N$ with $\frac{c_i}{u_i^E} = \frac{c_j}{u_j^E}$;
- *Claim monotonicity* if $f_i(E, c) \leq f_i(E, (c_i', c_{N\setminus\{i\}}))$ for all $i \in N$ with $c_i' > c_i$;

- *Truncation invariance* if $f(E, c) = f(E, \hat{c}^E)$, where
$$\hat{c}^E = \left(\min\{c_i, u_i^E\}\right)_{i \in N};$$
- *Conditional full compensation* if $f_i(E, c) = c_i$ for all $i \in N$ with
$$\left(\min\left\{\frac{c_i}{u_i^E}, \frac{c_j}{u_j^E}\right\} u_j^E\right)_{j \in N} \in E;$$
- *Independence of larger relative claims* if $f_i(E, c) = f_i(E, (c'_j, c_{N\setminus\{j\}}))$ for all $i, j \in N$ with $i \neq j$ and $\frac{c'_j}{u_j^E} > \frac{c_j}{u_j^E} \geq \frac{c_i}{u_i^E}$.

The constrained relative equal awards rule satisfies these five properties. The proportional rule and the constrained relative equal losses rule satisfy relative symmetry and claim monotonicity, but do not satisfy truncation invariance, conditional full compensation, and independence of larger relative claims. This is summarized in the following table.

	Prop	CREA	CREL
Relative symmetry	+	+	+
Claim monotonicity	+	+	+
Truncation invariance	−	+	−
Conditional full compensation	−	+	−
Independence of larger relative claims	−	+	−

Inspired by Dagan (1996), Yeh (2006), Herrero and Villar (2002), Yeh (2004) and Moulin and Shenker (1992), we generalize the axiomatic characterizations of the constrained relative equal awards rule in terms of relative symmetry, truncation invariance, and path composition up; claim monotonicity and conditional full compensation; conditional full compensation and path composition down; and relative symmetry and independence of larger relative claims to the domain of NTU-bankruptcy problems. The proof is postponed to the "Appendix".

Theorem 3 *The constrained relative equal awards rule is the unique bankruptcy rule satisfying*

(i) *Relative symmetry, truncation invariance, and path composition up;*
(ii) *Claim monotonicity and conditional full compensation;*
(iii) *Conditional full compensation and path composition down;*
(iv) *Relative symmetry and independence of larger relative claims.*

5 Concluding remarks

This paper introduces a duality notion for NTU-bankruptcy rules and derives several axiomatic characterizations of the proportional rule and the constrained relative equal awards rule. In follow-up research, axiomatic characterizations of the constrained relative equal losses rule can be obtained by applying a careful duality analysis. This is however not straightforward, since not all duality relations are inherited from the TU-bankruptcy model. In particular, Dietzenbacher et al. (2020) showed that the constrained relative equal losses rule does not satisfy *minimal rights first*, implying that minimal rights first and truncation invariance are not dual properties in the NTU-bankruptcy model. Similar challenges arise when defining a suitable generalization of the self-dual Talmud rule (cf. Aumann and Maschler 1985). Although our approach is successful in generalizing fundamental duality relations from the TU-bankruptcy model, we do not claim that our definition is the only interpretation of duality in the NTU-bankruptcy model. Possibly, alternative interpretations could serve alternative purposes. Moreover, NTU-bankruptcy problems could be further generalized to NTU-bankruptcy problems with a priori unions, in line with Borm et al. (2005), or to NTU-bankruptcy problems with references, in line with Pulido et al. (2002) and Pulido et al. (2008).

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Appendix

Proof of Theorem 2 (i) Let $(E, c) \in \mathrm{BR}^N$ and let $t \in (0, 1)$. Then

$$\mathrm{Prop}(tE, c) = \frac{1}{\lambda^{tE,c}} c = t \frac{1}{\lambda^{E,c}} c = t\mathrm{Prop}(E, c).$$

Hence, the proportional rule satisfies path linearity. Let f be a bankruptcy rule satisfying path linearity. Let $(E, c) \in \mathrm{BR}^N$. Then

$$f(E, c) = f\left(\frac{1}{\lambda^{E,c}} \lambda^{E,c} E, c\right) = \frac{1}{\lambda^{E,c}} f(\lambda^{E,c} E, c) = \frac{1}{\lambda^{E,c}} c = \mathrm{Prop}(E, c).$$

Hence, $f = \mathrm{Prop}$.

(ii) By Theorem 1(i), the proportional rule satisfies self-duality. By Theorem 2(i), the proportional rule satisfies path composition down. Let f be a bankruptcy rule satisfying path composition down and self-duality. Let $(E, c) \in \mathrm{BR}^N$. By path continuity, there is a $t \in (0, \lambda^{E,c})$ such that $\sum_{i \in N} f_i(tE, c) = \frac{1}{2} \sum_{i \in N} c_i$. By self-duality, $c - f(tE, c) = f(\lambda^{E,c-f(tE,c)} E, c)$. This means that

$$\sum_{i \in N} f_i\left(\lambda^{E,c-f(tE,c)} E, c\right) = \sum_{i \in N} c_i - \sum_{i \in N} f_i(tE, c) = \frac{1}{2} \sum_{i \in N} c_i.$$

By path monotonicity, $t = \lambda^{E,c-f(tE,c)}$ and $f(tE, c) = f(\lambda^{E,c-f(tE,c)} E, c)$. This means that $t = \frac{1}{2}\lambda^{E,c}$ and $f(tE, c) = \frac{1}{2}c$. Similarly, $f(\frac{1}{2}\lambda^{E,\frac{1}{2}c} E, \frac{1}{2}c) = \frac{1}{4}c$. By path composition down,

$$\begin{aligned} f\left(\tfrac{1}{4}\lambda^{E,c} E, c\right) &= f\left(\tfrac{1}{4}\lambda^{E,c} E, f\left(\tfrac{1}{2}\lambda^{E,c} E, c\right)\right) = f\left(\tfrac{1}{4}\lambda^{E,c} E, \tfrac{1}{2}c\right) \\ &= f\left(\tfrac{1}{2}\lambda^{E,\frac{1}{2}c} E, \tfrac{1}{2}c\right) = \tfrac{1}{4}c. \end{aligned}$$

By self-duality,

$$f\left(\tfrac{3}{4}\lambda^{E,c}E,c\right)=f\left(\lambda^{E,\frac{3}{4}c}E,c\right)=f\left(\lambda^{E,c-f\left(\frac{1}{4}\lambda^{E,c}E,c\right)}E,c\right)$$
$$=c-f\left(\tfrac{1}{4}\lambda^{E,c}E,c\right)=\tfrac{3}{4}c.$$

Continuing this reasoning, $f(\frac{m}{2^n}\lambda^{E,c}E,c)=\frac{m}{2^n}c$ for all $m,n\in\mathbb{N}$ with $m\leq 2^n$. By path continuity, $f(t\lambda^{E,c}E,c)=tc$ for all $t\in(0,1)$. Hence, $f(E,c)=\text{Prop}(E,c)$.

(iii) By Theorem 1(i), the proportional rule satisfies self-duality. By Theorem 2(i), the proportional rule satisfies path composition up. Let f be a bankruptcy rule satisfying path composition up and self-duality. By Lemma 2(ii), f satisfies path composition down and self-duality. By Theorem 2(ii), $f=\text{Prop}$.

Proof of Theorem 3 (i) Let $(E,c)\in\text{BR}^N$ and let $i,j\in N$ such that $\frac{c_i}{u_i^E}=\frac{c_j}{u_j^E}$. Then

$$\frac{\text{CREA}_i(E,c)}{u_i^E}=\frac{\min\{c_i,\alpha^{E,c}u_i^E\}}{u_i^E}=\min\left\{\frac{c_i}{u_i^E},\alpha^{E,c}\right\}$$
$$=\min\left\{\frac{c_j}{u_j^E},\alpha^{E,c}\right\}=\frac{\min\{c_j,\alpha^{E,c}u_j^E\}}{u_j^E}=\frac{\text{CREA}_j(E,c)}{u_j^E}.$$

Hence, the constrained relative equal awards rule satisfies relative symmetry. Let $(E,c)\in\text{BR}^N$. For all $i\in N$,

$$\text{CREA}_i(E,\hat{c}^E)=\min\{\hat{c}_i^E,\alpha^{E,\hat{c}^E}u_i^E\}=\min\{\min\{c_i,u_i^E\},\alpha^{E,\hat{c}^E}u_i^E\}$$
$$=\min\{c_i,u_i^E,\alpha^{E,\hat{c}^E}u_i^E\}=\min\{c_i,\alpha^{E,\hat{c}^E}u_i^E\}.$$

Since E is nonleveled, this means that $\text{CREA}(E,c)=\text{CREA}(E,\hat{c}^E)$. Hence, the constrained relative equal awards rule satisfies truncation invariance. Let $(E,c)\in\text{BR}^N$ and let $t\in(0,1)$. For all $i\in N$, $\text{CREA}_i(tE,c)=\min\{c_i,t\alpha^{tE,c}u_i^E\}$. This means that $\text{CREA}(tE,c)\leq\text{CREA}(E,c)$. Denote $d=\lambda^{E,\text{CREA}(E,c)-\text{CREA}(tE,c)}$. Suppose that $d\alpha^{dE,c-\text{CREA}(tE,c)}\leq\alpha^{E,c}-t\alpha^{tE,c}$. For all $i\in N$ with $\text{CREA}_i(tE,c)=c_i$,

$$\text{CREA}_i(dE,c-\text{CREA}(tE,c))=0=c_i-c_i=\text{CREA}_i(E,c)-\text{CREA}_i(tE,c).$$

For all $i\in N$ with $\text{CREA}_i(tE,c)=t\alpha^{tE,c}u_i^E$,

$$\begin{aligned}\text{CREA}_i(dE,c-\text{CREA}(tE,c))&=\min\left\{c_i-\text{CREA}_i(tE,c),\alpha^{dE,c-\text{CREA}(tE,c)}u_i^{dE}\right\}\\&=\min\left\{c_i-t\alpha^{tE,c}u_i^E,d\alpha^{dE,c-\text{CREA}(tE,c)}u_i^E\right\}\\&\leq\min\left\{c_i-t\alpha^{tE,c}u_i^E,\alpha^{E,c}u_i^E-t\alpha^{tE,c}u_i^E\right\}\\&=\min\{c_i,\alpha^{E,c}u_i^E\}-t\alpha^{tE,c}u_i^E\\&=\text{CREA}_i(E,c)-\text{CREA}_i(tE,c).\end{aligned}$$

Since E is nonleveled, $\text{CREA}(dE,c-\text{CREA}(tE,c))\in\text{P}(dE)$, and $\text{CREA}(E,c)-\text{CREA}(tE,c)\in\text{P}(dE)$, this means that

$$\text{CREA}(dE,c-\text{CREA}(tE,c))=\text{CREA}(E,c)-\text{CREA}(tE,c).$$

Clearly, similar arguments apply to the case $d\alpha^{dE,c-\mathrm{CREA}(tE,c)} > \alpha^{E,c} - t\alpha^{tE,c}$. Hence, the constrained relative equal awards rule satisfies path composition up. Let f be a bankruptcy rule satisfying relative symmetry, truncation invariance, and path composition up. Let $(E,c) \in \mathrm{BR}^N$. Suppose that $f(tE,c) \neq \mathrm{CREA}(tE,c)$ for some $t \in [0, \lambda^{E,c}]$. Define $\hat{t} = \inf\{t \in [0,\lambda^{E,c}] \mid f(tE,c) \neq \mathrm{CREA}(tE,c)\}$. By path continuity, $\hat{t} \in [0,\lambda^{E,c})$ and $f(\hat{t}E,c) = \mathrm{CREA}(\hat{t}E,c)$. Denote $N = \{1,\ldots,|N|\}$ such that $\frac{c_1}{u_1^E} \leq \cdots \leq \frac{c_{|N|}}{u_{|N|}^E}$. Let $k \in N$ be such that $f_i(\hat{t}E,c) = c_i$ for all $i < k$, and $f_i(\hat{t}E,c) = \hat{t}\alpha^{\hat{t}E,c}u_i^E < c_i$ for all $i \geq k$. Define $m = \min\{\|x\| \mid x \in \mathrm{P}(E)\}$. Let $\varepsilon \in (0, m(\frac{c_k}{u_k^E} - \frac{f_k(\hat{t}E,c)}{u_k^E}))$. By path continuity, there is a $\delta > 0$ such that $\|f(tE,c) - f(\hat{t}E,c)\| < \varepsilon$ for all $t \in (\hat{t}, \min\{\hat{t}+\delta, \lambda^{E,c}\})$. Let $t \in (\hat{t}, \min\{\hat{t}+\delta, \lambda^{E,c}\})$. By path monotonicity, $\lambda^{E,f(tE,c)-f(\hat{t}E,c)} \in (0,\lambda^{E,c})$. Denote $d = \lambda^{E,f(tE,c)-f(\hat{t}E,c)}$. By path composition up,

$$m\left(\frac{c_k}{u_k^E} - \frac{f_k(\hat{t}E,c)}{u_k^E}\right) > \varepsilon > \|f(tE,c) - f(\hat{t}E,c)\| = \|f(dE, c - f(\hat{t}E,c))\| \geq dm.$$

This means that $d < (\frac{c_k}{u_k^E} - \frac{f_k(\hat{t}E,c)}{u_k^E})$. Define $\tilde{u}^{dE} \in \mathbb{R}_+^N$ by

$$\tilde{u}_i^{dE} = \begin{cases} 0 & \text{for all } i < k; \\ u_i^{dE} & \text{for all } i \geq k. \end{cases}$$

For all $i < k$,

$$\tilde{u}_i^{dE} = 0 = c_i - c_i = c_i - f_i(\hat{t}E,c) = c_i - \mathrm{CREA}_i(\hat{t}E,c).$$

For all $i \geq k$,

$$\begin{aligned}\tilde{u}_i^{dE} = u_i^{dE} = du_i^E &< \left(\frac{c_k}{u_k^E} - \frac{f_k(\hat{t}E,c)}{u_k^E}\right)u_i^E \leq \left(\frac{c_i}{u_i^E} - \frac{\hat{t}\alpha^{\hat{t}E,c}u_k^E}{u_k^E}\right)u_i^E \\ &= c_i - \hat{t}\alpha^{\hat{t}E,c}u_i^E = c_i - f_i(\hat{t}E,c) = c_i - \mathrm{CREA}_i(\hat{t}E,c).\end{aligned}$$

By truncation invariance and relative symmetry,

$$\begin{aligned} f\big(dE, c - f(\hat{t}E,c)\big) = f\big(dE, \tilde{u}^{dE}\big) &= \frac{1}{\lambda^{dE,\tilde{u}^{dE}}}\tilde{u}^{dE} \\ &= \mathrm{CREA}\big(dE, \tilde{u}^{dE}\big) = \mathrm{CREA}\big(dE, c - \mathrm{CREA}(\hat{t}E,c)\big).\end{aligned}$$

By composition up,

$$\begin{aligned} f(tE,c) &= f(\hat{t}E,c) + f(dE, c - f(\hat{t}E,c)) \\ &= \mathrm{CREA}(\hat{t}E,c) + \mathrm{CREA}(dE, c - \mathrm{CREA}(\hat{t}E,c)) \\ &= \mathrm{CREA}(tE,c).\end{aligned}$$

This contradicts the definition of $\hat{t}$. Hence, $f(tE,c) = \mathrm{CREA}(tE,c)$ for all $t \in [0,\lambda^{E,c}]$.

(ii) Let $(E,c) \in \mathrm{BR}^N$ and let $i \in N$ with $c_i' > c_i$. If $\alpha^{E,(c_i',c_{N\setminus\{i\}})} \geq \alpha^{E,c}$, then

$$\mathrm{CREA}_i(E,(c_i',c_{N\setminus\{i\}})) = \min\{c_i', \alpha^{E,(c_i',c_{N\setminus\{i\}})}u_i^E\} \geq \min\{c_i, \alpha^{E,c}u_i^E\} = \mathrm{CREA}_i(E,c).$$

Suppose that $\alpha^{E,(c'_i,c_{N\setminus\{i\}})} < \alpha^{E,c}$. For all $j \in N \setminus \{i\}$,

$$\begin{aligned}\mathrm{CREA}_j(E,(c'_i,c_{N\setminus\{i\}})) &= \min\{c_j, \alpha^{E,(c'_i,c_{N\setminus\{i\}})}u^E_j\} \leq \min\{c_j, \alpha^{E,c}u^E_j\}\\ &= \mathrm{CREA}_j(E,c).\end{aligned}$$

Since E is nonleveled, this means that $\mathrm{CREA}_i(E,c') \geq \mathrm{CREA}_i(E,c)$. Hence, the constrained relative equal awards rule satisfies claim monotonicity. Let $(E,c) \in \mathrm{BR}^N$ and let $i \in N$ with

$$\left(\min\left\{\frac{c_i}{u^E_i}, \frac{c_j}{u^E_j}\right\}u^E_j\right)_{j\in N} \in E.$$

Suppose that $\mathrm{CREA}_i(E,c) = \alpha^{E,c}u^E_i$. For all $j \in N$,

$$\mathrm{CREA}_j(E,c) = \min\{c_j, \alpha^{E,c}u^E_j\} = \min\left\{\alpha^{E,c}, \frac{c_j}{u^E_j}\right\}u^E_j \leq \min\left\{\frac{c_i}{u^E_i}, \frac{c_j}{u^E_j}\right\}u^E_j.$$

Since E is nonleveled, this means that $\mathrm{CREA}_i(E,c) = c_i$. Hence, the constrained relative equal awards rule satisfies conditional full compensation. Let f be a bankruptcy rule satisfying claim monotonicity and conditional full compensation. Let $(E,c) \in \mathrm{BR}^N$. Let $i \in N$ with $\mathrm{CREA}_i(E,c) = c_i$. For all $j \in N$,

$$\min\left\{\frac{c_i}{u^E_i}, \frac{c_j}{u^E_j}\right\}u^E_j \leq \min\left\{\alpha^{E,c}, \frac{c_j}{u^E_j}\right\}u^E_j = \min\{c_j, \alpha^{E,c}u^E_j\} = \mathrm{CREA}_j(E,c).$$

Since E is comprehensive, this means that

$$\left(\min\left\{\frac{c_i}{u^E_i}, \frac{c_j}{u^E_j}\right\}u^E_j\right)_{j\in N} \in E.$$

By conditional full compensation, $f_i(E,c) = c_i$. Suppose that $f(E,c) \neq \mathrm{CREA}(E,c)$. Since E is nonleveled, there is a $k \in N$ with $f_k(E,c) < \mathrm{CREA}_k(E,c) = \alpha^{E,c}u^E_k < c_k$. Define $c'_k = \alpha^{E,c}u^E_k$. For all $j \in N$,

$$\min\left\{\frac{c'_k}{u^E_k}, \frac{c_j}{u^E_j}\right\}u^E_j = \min\left\{\alpha^{E,c}, \frac{c_j}{u^E_j}\right\}u^E_j = \min\{c_j, \alpha^{E,c}u^E_j\} = \mathrm{CREA}_j(E,c).$$

This means that

$$\left(\min\left\{\frac{c'_k}{u^E_k}, \frac{c_j}{u^E_j}\right\}u^E_j\right)_{j\in N} \in E.$$

By conditional full compensation, $f_k(E,(c'_k,c_{N\setminus\{k\}})) = c'_k$. Then $f_k(E,(c'_k,c_{N\setminus\{k\}})) > f_k(E,c)$, contradicting claim monotonicity. Hence, $f(E,c) = \mathrm{CREA}(E,c)$.

(iii) By Theorem 3(ii), the constrained relative equal awards rule satisfies conditional full compensation. Let $(E,c) \in \mathrm{BR}^N$ and let $t \in (0,1)$. For all $i \in N$, $\mathrm{CREA}_i(tE,c) = \min\{c_i, t\alpha^{tE,c}u^E_i\}$. This means that $\mathrm{CREA}(tE,c) \leq \mathrm{CREA}(E,c)$. Suppose that

$\alpha^{tE,\mathrm{CREA}(E,c)} \leq \alpha^{tE,c}$. For all $i \in N$,

$$\begin{aligned}\mathrm{CREA}_i(tE, \mathrm{CREA}(E,c)) &= \min\{\mathrm{CREA}_i(E,c), \alpha^{tE,\mathrm{CREA}(E,c)} u_i^{tE}\} \\ &\leq \min\{\min\{c_i, \alpha^{E,c} u_i^E\}, \alpha^{tE,c} u_i^{tE}\} \\ &= \min\{\min\{c_i, \alpha^{E,c} u_i^E\}, \min\{c_i, \alpha^{tE,c} u_i^{tE}\}\} \\ &= \min\{\mathrm{CREA}_i(E,c), \mathrm{CREA}_i(tE,c)\} \\ &= \mathrm{CREA}_i(tE,c).\end{aligned}$$

Since E is nonleveled, $\mathrm{CREA}(tE, \mathrm{CREA}(E,c)) \in \mathrm{P}(tE)$, and $\mathrm{CREA}(tE,c) \in \mathrm{P}(tE)$, this means that $\mathrm{CREA}(tE, \mathrm{CREA}(E,c)) = \mathrm{CREA}(tE,c)$. Clearly, similar arguments apply to the case $\alpha^{tE,\mathrm{CREA}(E,c)} > \alpha^{tE,c}$. Hence, the constrained relative equal awards rule satisfies path composition down. Let f be a bankruptcy rule satisfying conditional full compensation and path composition down. Let $(E,c) \in \mathrm{BR}^N$. Let $i \in N$ with $\mathrm{CREA}_i(E,c) = c_i$. For all $j \in N$,

$$\min\left\{\frac{c_i}{u_i^E}, \frac{c_j}{u_j^E}\right\} u_j^E \leq \min\left\{\alpha^{E,c}, \frac{c_j}{u_j^E}\right\} u_j^E = \min\{c_j, \alpha^{E,c} u_j^E\} = \mathrm{CREA}_j(E,c).$$

Since E is comprehensive, this means that

$$\left(\min\left\{\frac{c_i}{u_i^E}, \frac{c_j}{u_j^E}\right\} u_j^E\right)_{j \in N} \in E.$$

By conditional full compensation, $f_i(E,c) = c_i$. Suppose that $f(E,c) \neq \mathrm{CREA}(E,c)$. Since E is nonleveled, there is a $k \in N$ with $f_k(E,c) < \mathrm{CREA}_k(E,c) = \alpha^{E,c} u_k^E < c_k$. By path monotonicity and path continuity, there is a $t \in (1, \lambda^{E,c})$ such that $f_k(tE,c) = \alpha^{E,c} u_k^E$. For all $j \in N$,

$$\min\left\{\frac{f_k(tE,c)}{u_k^E}, \frac{f_j(tE,c)}{u_j^E}\right\} u_j^E \leq \min\left\{\alpha^{E,c}, \frac{c_j}{u_j^E}\right\} u_j^E = \mathrm{CREA}_j(E,c).$$

Since E is comprehensive, this means that

$$\left(\min\left\{\frac{f_k(tE,c)}{u_k^E}, \frac{f_j(tE,c)}{u_j^E}\right\} u_j^E\right)_{j \in N} \in E.$$

By conditional full compensation, $f_k(E, f(tE,c)) = f_k(tE,c)$. By path composition down, $f_k(E,c) = f_k(tE,c)$. This is a contradiction. Hence, $f(E,c) = \mathrm{CREA}(E,c)$.

(iv) By Theorem 3(i), the constrained relative equal awards rule satisfies relative symmetry. Let $(E,c) \in \mathrm{BR}^N$ and let $i, j \in N$ with $i \neq j$ and $\frac{c'_j}{u_j^E} > \frac{c_j}{u_j^E} \geq \frac{c_i}{u_i^E}$. Suppose that $\alpha^{E,(c'_j, c_{N\setminus\{j\}})} \geq \alpha^{E,c}$. For all $k \in N$,

$$\mathrm{CREA}_k(E, (c'_j, c_{N\setminus\{j\}})) \geq \min\{c_k, \alpha^{E,c} u_k^E\} = \mathrm{CREA}_k(E,c).$$

Since E is nonleveled, this means that $\mathrm{CREA}(E, (c'_j, c_{N\setminus\{j\}})) = \mathrm{CREA}(E,c)$. Now suppose that $\alpha^{E,(c'_j, c_{N\setminus\{j\}})} < \alpha^{E,c}$. For all $k \in N \setminus \{j\}$,

$$\mathrm{CREA}_k(E, (c'_j, c_{N\setminus\{j\}})) \leq \min\{c_k, \alpha^{E,c} u_k^E\} = \mathrm{CREA}_k(E,c).$$

Suppose that $\mathrm{CREA}_i(E, (c'_j, c_{N\setminus\{j\}})) < \mathrm{CREA}_i(E, c)$. Then $\alpha^{E,(c'_j, c_{N\setminus\{j\}})} < \frac{c_i}{u_i^E}$. Since E is nonleveled, $\mathrm{CREA}_j(E, (c'_j, c_{N\setminus\{j\}})) > \mathrm{CREA}_j(E, c)$. Then $\alpha^{E,(c'_j, c_{N\setminus\{j\}})} > \frac{c_j}{u_j^E}$. This is a contradiction. Hence, $\mathrm{CREA}_i(E, (c'_j, c_{N\setminus\{j\}})) = \mathrm{CREA}_i(E, c)$ and the constrained relative equal awards rule satisfies independence of larger relative claims. Let f be a bankruptcy rule satisfying relative symmetry and independence of larger relative claims. Let $(E, c) \in \mathrm{BR}^N$. Denote $N = \{1, \ldots, |N|\}$ such that $\frac{c_1}{u_1^E} \leq \cdots \leq \frac{c_{|N|}}{u_{|N|}^E}$. Let $k \in N$ be such that $\mathrm{CREA}_i(E, c) = c_i$ for all $i < k$, and $\mathrm{CREA}_i(E, c) = \alpha^{E,c} u_i^E < c_i$ for all $i \geq k$. For all $i < k$, by independence of larger relative claims,

$$f_i(E, c) = f_i(E, \mathrm{CREA}(E, c)) = \mathrm{CREA}_i(E, c).$$

For all $i \geq k$, define $c^i \in \mathbb{R}_+^N$ by

$$c_j^i = \begin{cases} c_j & \text{for all } j \leq i; \\ \frac{c_i}{u_i^E} u_j^E & \text{for all } j > i. \end{cases}$$

By independence of larger relative claims and relative symmetry,

$$f_k(E, c) = f_k(E, c^k) = \alpha^{E,c^k} u_k^E = \mathrm{CREA}_k(E, c^k) = \mathrm{CREA}_k(E, c).$$

Next, these arguments apply to claimant $k+1$. Continuing this reasoning, $f_i(E, c) = \mathrm{CREA}_i(E, c)$ for all $i \geq k$. Hence, $f(E, c) = \mathrm{CREA}(E, c)$.

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